# SPRING 2024 MATH 590: EXAM 2 SOLUTIONS

Name:

Throughout $V$ will denote a vector space over $F = \mathbb{R}$ or $\mathbb{C}$, $T$ a linear transformation from $V$ to $V$ and $A$ a matrix with entries in $F = \mathbb{R}$ or $\mathbb{C}$.

(I) **True-False:** Write true or false next to each of the statements below. (3 points each)

(a) Suppose $A$ is a $4 \times 4$ real matrix whose columns are linearly independent. Then the rows of $A$ are linearly independent. True. See the theorem from the Daily Update of February 28.

(b) Suppose $v_1, v_2 \in \mathbb{R}^2$ are eigenvectors for a real, symmetric $2 \times 2$ matrix. Then $v_1, v_2$ are orthogonal. False. $v_1$ and $v_2$ could belong to the same eigenspace.

(c) Let $\lambda \in \mathbb{R}$ be an eigenvalue of the matrix $A$. Then the geometric multiplicity of $A$ is less than or equal to the algebraic multiplicity of $A$. True. The geometric multiplicity is always less than or equal to the algebraic multiplicity. See Fact due from the Daily Update of March 18.

(d) Suppose $P$ and $A$ are $2 \times 2$ real matrices satisfying: $P^{-1} = P^t$ and $P^{-1}AP = D$, where $D$ is a diagonal matrix. Then $A$ is a symmetric matrix. True. From $P^tAP = D$, one gets $P^tA^tP = (P^tAP)^t = D^t = D = P^tAP$. Cancelling $P^t$ and $P$ gives $A^t = A$.

(e) Suppose $T : \mathbb{R}^4 \to \mathbb{R}^4$ is a symmetric linear transformation and $\alpha \subseteq \mathbb{R}^4$ is a basis. Then $[T]_\alpha^\alpha$ is a symmetric matrix. False. $\alpha$ must be an orthonormal basis.

(II) **State the indicated definition, proposition or theorem.** (5 points each)

(a) Suppose $\lambda \in F$ is an eigenvalue of $T : V \to V$. Define the *geometric multiplicity* and the *algebraic multiplicity* of $T$.

Solution. Suppose $p_T(x) = (x-\lambda)^e q(x)$, with $q(\lambda) \neq 0$. Then $e$ is the algebraic multiplicity of $\lambda$ and $\dim(E_\lambda)$ is the geometric multiplicity of $\lambda$.

(b) State the theorem characterizing when an $n \times n$ matrix $A$ with entries in $F$ is diagonalizable.

Solution. For an $n \times n$ matrix $A$, the following are equivalent:

(i) $A$ is diagonalizable.
(ii) $p_A(x) = (x - \lambda_1)^{e_1} \cdots (x - \lambda_r)^{e_r}$ and $\dim(E_{\lambda_i}) = e_i$, for $1 \leq i \leq r$.
(iii) $p_A(x) = (x - \lambda_1)^{e_1} \cdots (x - \lambda_r)^{e_r}$ and $\dim(E_{\lambda_1}) + \cdots + \dim(E_{\lambda_r}) = n$.

(c) State the theorem about eigenvectors associated to distinct eigenvalues.

Solution. Let $T : V \to V$ be a linear transformation with distinct eigenvalues $\lambda_1, \dots, \lambda_r$. If $v_1, \dots, v_r \in V$ are eigenvectors with $T(v_i) = \lambda_i v_i$, for $1 \le i \le r$, then $v_1, \dots, v_r$ are linearly independent.

(III) **Short Answer**. (15 points each)

(a) Give an example of a $3 \times 3$ matrix that has its eigenvalues in $\mathbb{R}$, but is **not** diagonalizable. You must justify your answer.

Solution. There are infinitely many matrices that satisfy these conditions. Take $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then $p_A(x) = (x-1)^3$, so $A$ has its eigenvalues in $\mathbb{R}$. Moreover 1 is an eigenvalue with algebraic multiplicity equal to three.

On the other hand, $E_1$ is the nullspace of $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, which has rank one. Therefore its nullspace has dimension two, i.e., $\dim(E_1) = 2 < 3$, so $A$ is not diagonalizable.

(b) Suppose $A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$. Give a detailed explanation why $A$ is diagonalizable. Note, we have $p_A(x) = x^4(x-5)$.

Solution. The eigenvalues of $A$ are 0, with algebraic multiplicity four, and 5 with algebraic multiplicity one.

$E_0$ is the nullspace of $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \xrightarrow{\text{EROs}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, which has rank one. Thus, the nullspace of $E_0$ has dimension four, which equals the algebraic multiplicity of 0.

On the other hand the algebraic multiplicity of 1 equals one, which forces the geometric multiplicity to be one, since the latter is less than or equal to the former. Therefore, $A$ is diagonalizable.

(c) Suppose $V$ is the vector space spanned by the matrices $v_1 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, v_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, v_3 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ over $\mathbb{R}$ with inner product $\langle A, B \rangle := \text{trace}(A^t B)$. Find an orthogonal basis for $V$.

Solution. We apply the Gram Schmidt process to $v_1, v_2, v_3$. Set $w_1 = v_1$. Then, $w_2 = v_2 \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1$. We have

$$\langle v_2, w_1 \rangle = \text{trace} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \right\} = \text{trace} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = 0,$$

so that $w_2 = v_2$.

$w_3 = v_3 - \frac{\langle v_3, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_3, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2$.

We have

$$\langle v_3, w_1\rangle = \text{trace}\left\{\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\cdot\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}\right\} = \text{trace}\begin{pmatrix}-1 & -1\\ -1 & -1\end{pmatrix} = -2$$
$$\langle w_1, w_1\rangle = \text{trace}\left\{\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}\cdot\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}\right\} = \text{trace}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} = 2$$
$$\langle v_3, w_2\rangle = \text{trace}\left\{\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right\} = \text{trace}\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix} = 0$$

Therefore,

$$w_3 = v_3 - \frac{-2}{2}w_1 = \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} + \begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

We now have an orthogonal basis for $V$, namely $w_1, w_2, w_3$.

(IV) **Proof Problem.** Suppose $A$ is a $7 \times 7$ matrix with entries in $\mathbb{R}$ with characteristic polynomial $p_A(x) = (x-\lambda_1)^2(x-\lambda_2)^3(x-\lambda_3)^2$ and $\dim(E_{\lambda_1}) = 2, \dim(E_{\lambda_2}) = 3, \dim(E_{\lambda_3}) = 2$. Give a **direct proof** that $A$ is diagonalizable and identify a matrix $P$ such that $P^{-1}AP$ is a diagonal matrix. Note, by a direct proof we mean one cannot use the theorem characterizing diagonalizability. (25 points)

Solution. Viewing $\mathbb{R}^7$ as the space of column vectors, let $u_1, u_2$ be a basis for $E_{\lambda_1}$, $v_1, v_2, v_3$ be a basis for $E_{\lambda_2}$, and $w_1, w_2$ be a basis for $E_{\lambda_3}$. Set $P = [u_1\ u_2\ v_2\ v_2\ v_3\ w_1\ w_2]$. From the definition of $P$, we have

$$\begin{aligned}AP &= [Au_1\ Au_2\ Av_1\ Av_2\ Av_3\ Aw_2\ Aw_n]\\ &= [\lambda_1 u_1\ \lambda_1 u_2\ \lambda_2 v_1\ \lambda_2 v_2\ \lambda_2 v_3\ \lambda_3 w_1\ \lambda_3 w_2]\\ &= PD,\end{aligned}$$

where $D$ is the $7 \times 7$ diagonal matrix with diagonal entries $\lambda_1, \lambda_1, \lambda_2, \lambda_2, \lambda_2, \lambda_3, \lambda_3$. Once we show that $u_1, u_2, v_1, v_2, v_3, w_1, w_2$ is a basis for $\mathbb{R}^7$, then $P$ is an invertible matrix. Thus, from $AP = PD$, we get $P^{-1}AP = D$, showing that $A$ is diagonalizable. But seven linearly independent vectors in $\mathbb{R}^7$ form a basis for $\mathbb{R}^7$, so we just have to show that the columns of $P$ are linearly independent.

Suppose

$$\alpha_1 u_1 + \alpha_2 u_2 + \beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3 + \gamma_1 w_1 + \gamma_2 w_2 = \vec{0}.$$

If we set $A := \alpha_1 u_1 + \alpha_2 u_2, B := \beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3$, and $C := \gamma_1 w_1 + \gamma_2 w_2$, then $A \in E_{\lambda_1}, B \in E_{\lambda_2}, C \in E_{\lambda_3}$. We also have

$$1 \cdot A + 1 \cdot B + 1 \cdot C = \vec{0}.$$

Since eigenvectors corresponding to distinct eigenvalues are linearly independent, it follows that we must have $A = \vec{0}, B = \vec{0}, C = \vec{0}$. But $v_1, v_2$ are linearly independent, so $A = \vec{0}$ implies $\alpha_1 = \alpha_2 = 0$. Similarly, $v_1, v_2, v_3$ are linearly independent, so we have $\beta_1 = \beta_2 = \beta_3 = 0$, and likewise, $\gamma_1 = \gamma_2 = 0$, which shows that $u_1, u_2, v_1, v_2, v_3, w_1, w_2$ are linearly independent.

**Bonus Problems.** For ten bonus points, solve **one, and only one**, of the following bonus problems. In order to receive any bonus points, your answer must be completely (or, very close to completely) correct.

1. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a symmetric linear transformation. Prove that $[T]^\alpha_\alpha$ is a symmetric matrix, for every orthonormal basis $\alpha \subseteq \mathbb{R}^2$. Give an example where this fails, if $\alpha$ is not an orthonormal basis.

Solution. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a symmetric linear transformation. Prove that $[T]^\alpha_\alpha$ is a symmetric matrix, for every orthonormal basis $\alpha \subseteq \mathbb{R}^2$. Give an example where this fails, if $\alpha$ is not an orthonormal basis.

Solution. Suppose $\alpha = \{u_1, u_2\}$ is an orthonormal basis for $\mathbb{R}^2$ and $T(u_1) = au_1 + bu_2$, $T(u_2) = cu_1 + du_2$. It follows that $[T]^\alpha_\alpha = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$.

We also have

$$T(u_1) \cdot u_2 = (au_1 + bu_2) \cdot u_2 = a(u_1 \cdot u_2) + b(u_2 \cdot u_2) = a0 + b1 = b,$$

and moreover,

$$u_1 \cdot T(u_2) = u_1 \cdot (cu_1 + du_2) = c(u_1 \cdot u_1) + d(u_1 \cdot u_2) = c1 + d0 = c.$$

Since $T(u_1) \cdot u_2 = u_1 \cdot T(u_2)$, it follows that $b = c$, showing that $[T]^\alpha_\alpha$ is symmetric.

Now consider $T(x, y) = (x + 2y, 2x + y)$, a symmetric linear transformation. If we let $v_1 = (1, 1)$ and $v_1 = (1, 0)$, then $\beta = \{v_1, v_2\}$ is a basis for $\mathbb{R}^2$ (since the corresponding determinant is not zero). On the other hand, $T(v_1) = (3, 3) = 3 \cdot v_1 + 0 \cdot v_2$ and $T(v_2) = (1, 2) = 2 \cdot v_1 - 1 \cdot v_2$, so that $[T]^\beta_\beta = \begin{pmatrix} 3 & 2 \\ 0 & -1 \end{pmatrix}$, which is not a symmetric matrix.

2. An important fact in linear algebra is the following: Suppose $A$ and $B$ are $n \times n$ diagonalizable matrices. If $AB = BA$, then $A$ and $B$ are *simultaneously diagonalizable*, i.e., there exists an invertible $n \times n$ matrix $P$ such that $P^{-1}AP$ and $P^{-1}BP$ are diagonal matrices. For the matrices $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix}$, verify that $AB = BA$ and find a $2 \times 2$ invertible matrix $P$ such that $P$ simultaneously diagonalizes both $A$ and $B$.

Solution. $AB = \begin{pmatrix} 10 & 11 \\ 11 & 10 \end{pmatrix} = BA$. Moreover, $p_A(x) = (x - 1)(x - 3)$ and $p_B(x) = (x + 1)(x - 7)$.

For $A$, we have $E_1$ is the null space of $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ which has basis $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, while $E_3$ is the null psace of $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ which has basis $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Thus, for $P = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, we have $P^{-1}AP = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$.

For $B$,we have $E_{-1}$ is the null space of $\begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}$, which has basis $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $E_7$ is the null space of $\begin{pmatrix} -4 & 4 \\ 4 & -4 \end{pmatrix}$ which has basis $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, thus, for $P = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, we have $P^{-1}BP = \begin{pmatrix} -1 & 0 \\ 0 & 7 \end{pmatrix}$, which shows that $P$ diagonalizes both $A$ and $B$.